

Independent random variables on Abelian groups with independent the sum and difference

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Abstract

Let X be a second countable locally compact Abelian group. Let ξ_1, ξ_2 be independent random variables with values in the group X and distributions μ_1, μ_2 such that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent. Assuming that the connected component of zero of the group X contains a finite number elements of order 2 we describe the possible distributions μ_k .

Key words. Locally compact Abelian group, Kac–Bernstein theorem, Gaussian measure.

1. Introduction. The classical Kac–Bernstein theorem states:

Theorem A. *Let ξ_1, ξ_2 be independent random variables. If the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent, then the random variables ξ_k are Gaussian.*

Much research has been devoted to group analogues of this theorem (see [1] – [12]). In the present article we study the following question: Let ξ_1, ξ_2 be independent random variables with values in a locally compact Abelian group. Assume that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent. What one can say about distributions of the random variables ξ_k . Before we pass to the proof of the main result recall some necessary notation and definitions.

Let X be a second countable locally compact Abelian group. Denote by $Y = X^*$ the character group of the group X , and by (x, y) the value of a character $y \in Y$ at a point $x \in X$. Denote by $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ the circle group (the one-dimensional torus). Denote by $M^1(X)$ the convolution semigroup of probability distributions on X . The characteristic function of a distribution $\mu \in M^1(X)$ we define by the formula

$$\hat{\mu}(y) = \int_X (x, y) d\mu(x).$$

A distribution $\gamma \in M^1(X)$ is called Gaussian ([13]), if its characteristic function is represented in the form

$$\hat{\gamma}(y) = (x, y) \exp\{-\varphi(y)\}, \quad y \in Y, \quad (1)$$

where $x \in X$ and $\varphi(y)$ is a continuous nonnegative function on Y satisfying the equation

$$\varphi(u + v) + \varphi(u - v) = 2[\varphi(u) + \varphi(v)], \quad u, v \in Y. \quad (2)$$

Denote by $\Gamma(X)$ the set of Gaussian distributions on the group X . A Gaussian distribution γ is called symmetric if in (1) $x = 0$. Denote by $\Gamma^s(X)$ the set of symmetric Gaussian distributions on X . Denote by $I(X)$ the set of idempotent distributions on the group X , i.e. the set of shifts of Haar distributions m_K of compact subgroups K of the group X .

We will formulate now the following problem.

Problem 1. Let X be a second countable locally compact Abelian group. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2 . Assume that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent. Describe the possible distributions μ_k .

A.L. Rukhin in [1] and [2] received some sufficient conditions for the group X in order that distributions μ_1 and μ_2 were represented as convolutions of Gaussian and idempotent distributions. The complete descriptions of such groups has been obtained by the author.

Theorem B ([4]), see also ([14, §7]). *Let X be a second countable locally compact Abelian group. Assume that the connected component of zero of the group X contains no elements of order 2. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2 such that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent. Then $\mu_k \in \Gamma(X) * I(X)$, $k = 1, 2$, and $\mu_1 = \mu_2 * E_x$, $x \in X$.*

*If the connected component of zero of a group X contains elements of order 2, then there exist independent random variables ξ_1, ξ_2 with values in X and distributions λ_1, λ_2 such that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent, but $\lambda_k \notin \Gamma(X) * I(X)$, $k = 1, 2$.*

We note that if a distribution $\mu \in \Gamma(X) * I(X)$, then μ is invariant with respect to a compact subgroup K , and μ induces a Gaussian distribution on the factor-group X/K under the natural homomorphism $X \mapsto X/K$.

Problem 1 was solved in [5] for the group $X = \mathbf{T}$, and was solved in [11] for the group $X = \mathbf{R} \times \mathbf{T}$ and \mathbf{a} -adic solenoids $\Sigma_{\mathbf{a}}$. Taking into account Theorem B solution of Problem 1 is reduced to the description of distributions μ_1 and μ_2 for groups X which contain elements of order 2. In the present article we solve Problem 1 for groups X satisfying the following condition: the connected component of zero of the group X contains a finite number of elements of order 2. Note that if this condition holds, under additional assumption that the random variables ξ_1 and ξ_2 are identically distributed Problem 1 was solved in [12].

We need some results about the structure of locally compact Abelian groups and the duality theory (see [15, Ch. 6]). If G is a closed subgroup of X , then denote by $A(Y, G) = \{y \in Y : (x, y) = 1 \text{ for all } x \in G\}$ its annihilator. The factor-group $Y/A(Y, G)$ is topologically isomorphic to the character group of the group G . Put $X_{(n)} = \{x \in X : nx = 0\}$, $X^{(n)} = \{x \in X : x = n\tilde{x}, \tilde{x} \in X\}$. A group X is said to be Corwin group if $X^{(2)} = X$. Denote by $\overline{Y^{(2)}}$ the closure of the subgroup $Y^{(2)}$. Consider the subgroup $X_{(2)}$. It is obvious that $A(Y, X_{(2)}) = \overline{Y^{(2)}}$. Assume that $X_{(2)}$ is a finite subgroup, and let n be the number of its elements $|X_{(2)}| = n$. Then $X_{(2)} \cong (X_{(2)})^* \cong \overline{Y^{(2)}}$. Let $Y = \bigcup_{j=0}^{n-1} (y_j + \overline{Y^{(2)}})$ be a decomposition of the group Y with respect to the subgroup $\overline{Y^{(2)}}$. If G is a subgroup of $X_{(2)}$, then its annihilator $H = A(Y, G)$, as is easily seen, is of the form $H = \bigcup_{j=0}^{l-1} (y_j + \overline{Y^{(2)}})$. Denote by c_X the connected component of zero of the group X . Denote by \mathbf{Z} the group of integers, and by $\mathbf{Z}(m) = \{0, 1, \dots, m-1\}$ the group of residue classes modulo m .

Let $f(y)$ be an arbitrary function on the group Y , and let $h \in Y$. Denote by Δ_h the finite difference operator

$$\Delta_h f(y) = f(y + h) - f(y), \quad y \in Y.$$

Let K be a compact subgroup of the group X . Then the characteristic function of the Haar distribution m_K is of the form

$$\hat{m}_K(y) = \begin{cases} 1, & y \in A(Y, K), \\ 0, & y \notin A(Y, K). \end{cases}$$

Denote by E_x the degenerate distribution concentrated at a point $x \in X$. For $\mu \in M^1(X)$ denote by $\sigma(\mu)$ the support of μ .

2. Solution of Problem 1. Let ξ_1, ξ_2 be independent random variables with values in the group X and with distributions μ_1, μ_2 . It is easily seen that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent if and only if the characteristic functions $\hat{\mu}_k(y) = \mathbf{E}[(\xi_k, y)]$ satisfy the equation

$$\hat{\mu}_1(u + v)\hat{\mu}_2(u - v) = \hat{\mu}_1(u)\hat{\mu}_2(u)\hat{\mu}_1(v)\hat{\mu}_2(-v), \quad u, v \in Y. \quad (3)$$

In the sequel we need the following lemmas.

Lemma 1 ([4], see also ([14, §9]) *Let ξ_1 and ξ_2 be independent random variables with values in a group X and distributions μ_1 and μ_2 . If the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent, then distributions μ_k can be replaced by their shifts μ'_k in such a manner that $\sigma(\mu'_k) \subset M$, $k = 1, 2$, where M is a subgroup of X such that M is topologically isomorphic to a group of the form $\mathbf{R}^m \times K$, where $m \geq 0$ and K is a compact Corwin group.*

Lemma 2 ([12]). *Let a locally compact Abelian group X be of the form $X = \mathbf{R}^m \times K$, where $m \geq 0$ and K is a compact Corwin group. Then $\overline{Y^{(2)}} = Y^{(2)}$ and $X_{(2)} \subset c_X$.*

Lemma 3 ([12]). *Let X be a finite Abelian group. Then for any function $f(y)$ on the group Y there is a complex measure δ on X such that $\delta(y) = f(y)$, $y \in Y$. If all nonzero elements of X have order 2, and $f(y)$ is a real-valued function, then δ is a signed measure.*

Lemma 4 ([16]). *Let Y be a locally compact Abelian group, $\psi(y)$ be a continuous function on Y satisfying the equation*

$$\Delta_h^2 \Delta_{2k} \psi(y) = 0, \quad h, k, y \in Y,$$

and the conditions $\psi(-y) = \psi(y)$, $\psi(0) = 0$. Let

$$Y = \bigcup_{\alpha} (y_{\alpha} + \overline{Y^{(2)}})$$

be the decomposition of the group Y with respect to the subgroup $\overline{Y^{(2)}}$. Then the function $\psi(y)$ can be represented in the form

$$\psi(y) = \varphi(y) + r_{\alpha}, \quad y \in y_{\alpha} + \overline{Y^{(2)}},$$

where $\varphi(y)$ is a continuous function on Y satisfying equation (2).

The main result of the article is the following theorem.

Theorem 1. *Let X be a second countable locally compact Abelian group. Assume that the connected component of zero of the group X contains a finite number elements of order 2. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2 such that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent. Then the following statements hold.*

1. *There exists a subgroup $G \subset X_{(2)}$ such that distributions $p(\mu_k)$ (p is the natural homomorphism $p : X \mapsto X/G$) are of the form:*

$$p(\mu_k) = \gamma * \pi_k * m_V * E_{x_k},$$

where $\gamma \in \Gamma^s(X/G)$, π_k are signed measures on $(X/G)_{(4)}$, V is a compact Corwin subgroup of the factor-group X/G , $x_k \in X/G$.

2. *If a group X is topologically isomorphic to a group of the form $\mathbf{R}^m \times K$, where $m \geq 0$ and K is a compact Corwin group, then either $\hat{\mu}_1(y) \equiv 0$ or $\hat{\mu}_2(y) \equiv 0$ for each coset $y_j + Y^{(2)}$ which disjoint with $A(Y, G)$.*

Proof. By Lemma 1 the distributions μ_k can be replaced with their shifts μ'_k in such a manner that $\sigma(\mu'_k) \subset M$, where M is a subgroup of X such that M is topologically isomorphic to a group of the form $\mathbf{R}^m \times K$, where $m \geq 0$ and K is a compact Corwin group. Thus, we may assume from the beginning that the group X is of the mentioned form. Then by Lemma 2 $X_{(2)} \subset c_X$, and hence $X_{(2)}$ is a finite subgroup. We note that the character group Y is topologically isomorphic to a group of the form $\mathbf{R}^m \times D$, where D is a discrete group without elements of order 2. It is obvious that $\overline{Y^{(2)}} = Y^{(2)}$. Let $|X_{(2)}| = n$, and

$$Y = \bigcup_{j=0}^{n-1} (y_j + Y^{(2)}) \tag{4}$$

be a decomposition of the group Y with respect to the subgroup $Y^{(2)}$. Put $N_k = \{y \in Y : \widehat{\mu}_k(y) \neq 0\}$, $k = 1, 2$, $N = N_1 \cap N_2$. It follows from equation (3) that N is an open subgroup of Y satisfying the following conditions: if $2y \in N$, then $y \in N$, and

$$N \cap Y^{(2)} = N^{(2)}. \quad (5)$$

Equation (3) also implies that

$$|\widehat{\mu}_1(u+v)| |\widehat{\mu}_2(u-v)| = |\widehat{\mu}_1(u-v)| |\widehat{\mu}_2(u+v)|$$

for all $u, v \in Y$. It follows from this that for arbitrary elements a and b from a given coset $y_j + Y^{(2)}$ we have the equality

$$|\widehat{\mu}_1(a)| |\widehat{\mu}_2(b)| = |\widehat{\mu}_1(b)| |\widehat{\mu}_2(a)|. \quad (6)$$

Denote by H a union of cosets $y_j + Y^{(2)}$ such that $N \cap (y_j + Y^{(2)}) \neq \emptyset$. Since N is a subgroup of Y , we conclude that H is also a subgroup of Y . Changing if it is necessary the numeration, we can assume that

$$H = \bigcup_{j=0}^{l-1} (y_j + Y^{(2)}). \quad (7)$$

Moreover, we may assume that $y_j \in N$. Note also that (5) implies that

$$N = \bigcup_{j=0}^{l-1} (y_j + N^{(2)}). \quad (8)$$

Put $G = A(X, H)$. Then $H^* \cong X/G$ and $H = A(Y, G)$. It is clear that $G \subset X_{(2)}$. Assume that $N \cap (y_j + Y^{(2)}) = \emptyset$ for some j . If $a \in N_1 \cap (y_j + Y^{(2)})$, then $a \notin N_2 \cap (y_j + Y^{(2)})$, and the right-hand side of (6) vanishes. Hence, $\widehat{\mu}_2(b) = 0$ for any $b \in y_j + Y^{(2)}$. Thus, we proved statement 2 of Theorem 1. This reasoning also shows that if $N \cap (y_j + Y^{(2)}) \neq \emptyset$, then $N_1 \cap (y_j + Y^{(2)}) = N_2 \cap (y_j + Y^{(2)})$.

Consider the restriction of equation (3) to N . It is obvious that the functions $|\widehat{\mu}_1(y)|$ and $|\widehat{\mu}_2(y)|$ also satisfy equation (3). Put $f_k(y) = -\ln |\widehat{\mu}_k(y)|$, $y \in N$, $k = 1, 2$. It follows from (3) that the functions $f_k(y)$ satisfy the equation

$$f_1(u+v) + f_2(u-v) = A(u) + A(v), \quad u, v \in N, \quad (9)$$

where $A(u) = f_1(u) + f_2(u)$. Apply the finite difference method to solve equation (9). Let h be an arbitrary element of N . Substitute $u+h$ for u and $v+h$ for v in equation (9). Subtracting equation (9) from the resulting equation we obtain

$$\Delta_{2h} f_1(u+v) = \Delta_h A(u) + \Delta_h A(v), \quad u, v \in N. \quad (10)$$

Put in (10) $v = 0$ and subtract from (10) the obtained equation. We get

$$\Delta_v \Delta_{2h} f_1(u) = \Delta_h A(v) + \Delta_h A(0), \quad u, v \in N.$$

It follows from this that

$$\Delta_v^2 \Delta_{2h} f_1(u) = 0, \quad u, v, h \in N.$$

The function $f_2(y)$ satisfies the same equation. Applying Lemma 4 we obtain the representations:

$$f_k(y) = \varphi_k(y) + p_{k,j}, \quad y \in y_j + N^{(2)}, \quad k = 1, 2.$$

Since equation (3) implies that $|\widehat{\mu}_1(2y)| = |\widehat{\mu}_2(2y)|$, $y \in Y$, the functions $\varphi_1(y)$ and $\varphi_2(y)$ coincide on $N^{(2)}$ and hence, they also coincide on N , i.e.

$$\varphi_1(y) = \varphi_2(y) = \varphi(y), \quad y \in N.$$

Taking this into account, we get from (3) that $p_{1,j} = -p_{2,j} = p_j$, $j = 0, 1, \dots, l-1$. It is obvious that $\varphi(y) \geq 0$.

Put $l_k(y) = \widehat{\mu}_k(y)/|\widehat{\mu}_k(y)|$, $y \in N$, $k = 1, 2$. The functions $l_1(y)$ and $l_2(y)$ satisfy the equation

$$l_1(u+v)l_2(u-v) = l_1(u)l_2(u)l_1(v)l_2(-v), \quad u, v \in N. \quad (11)$$

Moreover, $|l_k(u)| = 1$, $l_k(-u) = \overline{l_k(u)}$, $u \in N$, $l_k(0) = 1$, $k = 1, 2$. Putting in (11) $v = u$, and then $v = -u$, we find

$$l_k(2u) = l_k^2(u), \quad u \in N, \quad k = 1, 2. \quad (12)$$

Replacing in equation (11) u by v we get

$$l_1(u+v)l_2(v-u) = l_1(v)l_2(v)l_1(u)l_2(-u), \quad u, v \in N. \quad (13)$$

Multiplying (11) and (13) we find

$$l_1^2(u+v) = l_1^2(u)l_1^2(v), \quad u, v \in N. \quad (14)$$

It follows from (12) and (14) that the restriction of the function $l_1(y)$ to $N^{(2)}$ is a character of the group $N^{(2)}$. The same reasoning shows that the restriction of the function $l_2(y)$ to $N^{(2)}$ is also a character of the group $N^{(2)}$. Extend these characters from $N^{(2)}$ to some characters of the group H . By the duality theorem there exist elements $x_k \in X/G$ such that $l_k(y) = (x_k, y)$, $y \in N^{(2)}$. Put $l'_k(y) = (-x_k, y)l_k(y)$, $y \in H$. Then

$$l'_k(2y) = 1, \quad y \in N, \quad k = 1, 2, \quad (15)$$

and (12) implies that $l'_k(y) = \pm 1$, $y \in N$, $k = 1, 2$. Hence,

$$l'_k(y) = l'_k(-y), \quad y \in N, \quad k = 1, 2, \quad (16)$$

and the functions $l'_1(y)$ and $l'_2(y)$ satisfy the equation

$$l'_1(u+v)l'_2(u-v) = l'_1(u)l'_2(u)l'_1(v)l'_2(v), \quad u, v \in N. \quad (17)$$

This implies that for any elements a and b from a given coset $y_j + N^{(2)}$ the equality

$$l'_1(a)l'_2(b) = l'_1(b)l'_2(a) \quad (18)$$

holds. Fix an element $a \in y_j + N^{(2)}$. It follows from (18) that for all $b \in y_j + N^{(2)}$ either $l'_1(b) \equiv l'_2(b)$ or $l'_1(b) \equiv -l'_2(b)$. Consider the subgroup

$$L = N^{(2)} \cup (y_j + N^{(2)}),$$

where $y_j \in N$. Taking into account that $l'_1(y) \equiv l'_2(y)$ for $y \in N^{(2)}$, and for $y \in y_j + N^{(2)}$ either $l'_1(y) \equiv l'_2(y)$ or $l'_1(y) \equiv -l'_2(y)$, it easily follows from (17) that

$$l'_k(u+v)l'_k(u-v) = 1, \quad u, v \in L, \quad k = 1, 2. \quad (19)$$

We conclude now from (8) and (19) that

$$l'_k(u + 4v) = l'_k(u), \quad u, v \in N, \quad k = 1, 2, \quad (20)$$

i.e. the functions $l'_k(y)$ are invariant with respect to the subgroup $N^{(4)}$.

It follows from representation (4) that $Y^{(2)} = \bigcup_{j=0}^{n-1} (2y_j + Y^{(4)})$. Taking into consideration (7) this implies that

$$H = \bigcup_{0 \leq i \leq n-1, 0 \leq j \leq l-1} (2y_i + y_j + Y^{(4)}). \quad (21)$$

Similarly, (8) implies that

$$N = \bigcup_{0 \leq i, j \leq l-1} (2y_i + y_j + N^{(4)}).$$

Taking into account (21), extend the functions $l'_k(y)$ from N to some functions $\tilde{l}'_k(y)$ on H by the formulas

$$\begin{aligned} \tilde{l}'_k(2y_i + y_j + u) &= l'_k(2y_i + y_j), \quad u \in Y^{(4)}, \quad 0 \leq i, j \leq l-1, \\ \tilde{l}'_k(2y_i + y_j + u) &= 1, \quad u \in Y^{(4)}, \quad l \leq i \leq n-1, \quad 0 \leq j \leq l-1, \quad k = 1, 2. \end{aligned}$$

It follows from (16) and (21) that the functions $\tilde{l}'_k(y)$ also satisfy the condition

$$\tilde{l}'_k(y) = \tilde{l}'_k(-y), \quad y \in H, \quad k = 1, 2. \quad (22)$$

By construction the functions $\tilde{l}'_k(y)$ are invariant with respect to the subgroup $Y^{(4)}$, in particular they satisfy the equation

$$\tilde{l}'_k(u + 4v) = \tilde{l}'_k(u), \quad u, v \in H, \quad k = 1, 2, \quad (23)$$

and hence, they define some functions on the factor-group $H/Y^{(4)}$. Set $F = A(X/G, Y^{(4)}) \subset (X/G)_{(4)}$ and note that $F \cong (H/Y^{(4)})^*$. Taking into account that $H/Y^{(4)}$ is a finite group and applying Lemma 3 we conclude that there exist complex measures δ_k on X/G supported in the group F such that $\widehat{\delta}_k(y) = \tilde{l}'_k(y)$, $y \in H$. It is obvious that $F \cong (\mathbf{Z}(4))^m$ for some m . An arbitrary character of the group $(\mathbf{Z}(4))^m$ is of the form

$$((k_1, \dots, k_m), (l_1, \dots, l_m)) = \exp \left\{ \frac{i\pi}{2} \sum_{j=1}^m k_j l_j \right\},$$

$(k_1, \dots, k_m) \in (\mathbf{Z}(4))^m$, $(l_1, \dots, l_m) \in ((\mathbf{Z}(4))^m)^*$. The complex measures δ_k , considering as complex measures on F , are defined by the formulas

$$\delta_k\{(k_1, \dots, k_m)\} = \frac{1}{4^m} \sum_{(l_1, \dots, l_m) \in ((\mathbf{Z}(4))^m)^*} \tilde{l}'_k(l_1, \dots, l_m) \exp \left\{ \frac{i\pi}{2} \sum_{j=1}^m k_j l_j \right\}, \quad k = 1, 2. \quad (24)$$

Note that (22) and (23) imply that

$$\tilde{l}'_k(3y) = \tilde{l}'_k(y), \quad y \in H, \quad k = 1, 2. \quad (25)$$

We also note that $\exp \left\{ \frac{i\pi}{2} \sum_{j=1}^m k_j l_j \right\} = \pm i$ if and only if when $\sum_{j=1}^m k_j l_j$ is an odd number. Since (25) implies that

$$\tilde{l}'_k(l_1, \dots, l_m) = \tilde{l}'_k(3l_1, \dots, 3l_m), \quad (26)$$

and the numbers $\exp\left\{\frac{i\pi}{2}\sum_{j=1}^m k_j l_j\right\}$ and $\exp\left\{\frac{i\pi}{2}\sum_{j=1}^m 3k_j l_j\right\}$ are complex conjugate, it follows from (24) that all numbers $\delta_k\{(k_1, \dots, k_m)\}$ are real, i.e. actually δ_k are signed measures.

Return to the representations of the functions $f_k(y)$ on N . We have

$$f_1(y) = \varphi(y) + p_j, \quad f_2(y) = \varphi(y) - p_j, \quad y \in y_j + N^{(2)}, \quad j = 0, 1, \dots, l-1.$$

By Lemma 3, there exist signed measures ϵ_1 and ϵ_2 on $(X/G)_{(2)}$ such that

$$\hat{\epsilon}_1(y) = e^{p_j}, \quad \hat{\epsilon}_2(y) = e^{-p_j}, \quad y \in y_j + Y^{(2)}, \quad j = 0, 1, \dots, l-1. \quad (27)$$

Put $\pi_k = \delta_k * \epsilon_k$, $k = 1, 2$. Then π_k are signed measures on $(X/G)_{(4)}$. Extend the function $\varphi(y)$ from N to H retaining its properties ([17, §5.2]). Denote by $\tilde{\varphi}(y)$ the extended function. Let γ be a symmetric Gaussian distribution on the factor-group X/G with the characteristic function $\hat{\gamma}(y) = \exp\{-\tilde{\varphi}(y)\}$, $y \in H$. Set $V = A(X/G, N)$. It is easily seen that V is a compact Corwin subgroup. Thus, we obtained that the restriction to H of the characteristic functions of the distributions μ_k can be represented in the form

$$\hat{\mu}_k(y) = \begin{cases} \exp\{-\varphi(y)\} \hat{\pi}_k(y)(x_k, y), & y \in N, \\ 0, & y \in H \setminus N, \end{cases}$$

$k = 1, 2$. The desired representation $p(\mu_k) = \gamma * \pi_k * m_V * E_{x_k}$, $k = 1, 2$, results now from the following general remark: if $\mu \in M^1(X)$, and H is a closed subgroup of Y , then the restriction of the characteristic function $\hat{\mu}(y)$ to H is the characteristic function of the distribution $p(\mu)$ on the factor-group X/G , where $G = A(X, H)$, and p is the natural homomorphism $p : X \mapsto X/G$. The theorem is proved completely.

3. Some remarks. Give some comments to Theorem 1.

Remark 1. Let X be a second countable locally compact Abelian group such that X is topologically isomorphic to a group of the form $\mathbf{R}^m \times K$, where $m \geq 0$ and K is a compact Corwin group. Assume also that X contains only one element of order 2, i.e. $X_{(2)} \cong \mathbf{Z}(2)$. Then we can strengthen Theorem 1. We reason as in the proof of Theorem 1 and retain the same notation.

Since $X_{(2)}$ is a finite group and $Y^{(2)} = A(Y, X_{(2)})$, we have $X_{(2)} \cong (X_{(2)})^* \cong Y/Y^{(2)}$. Hence, a decomposition of the group Y with respect to the subgroup $Y^{(2)}$ is of the form $Y = Y^{(2)} \cup (y_1 + Y^{(2)})$, and there are two possibilities for the subgroup H : either $H = Y$, and then $G = A(X, Y) = \{0\}$, or $H = Y^{(2)}$, and then $G = A(X, Y^{(2)}) = X_{(2)}$.

1. Let $H = Y$, then

$$N = N^{(2)} \cup (y_1 + N^{(2)}). \quad (28)$$

Since $l'_1(y) = l'_2(y)$ for $y \in N^{(2)}$ and $l'_1(y) = \pm l'_2(y) = \pm 1$ for $y \in y_1 + N^{(2)}$, it follows from (28) that the functions $l'_k(y)$ are characters of the subgroup N . Extend these characters from N to some characters of the group Y . By the duality theorem there exist elements $t_k \in X$ such that $l'_k(y) = (t_k, y)$, $y \in N$. Set $z_k = x_k + t_k$. We obtain as a result the representation

$$\mu_k = \gamma * \epsilon_k * m_V * E_{z_k}, \quad k = 1, 2,$$

where $\gamma \in \Gamma^s(X)$, ϵ_k are signed measures on $X_{(2)}$, V is a compact Corwin subgroup of the group X , $z_k \in X$. Moreover, it follows from (27) that such that $\epsilon_1 * \epsilon_2 = E_0$.

2. Let $H = Y^{(2)}$. Then $N \subset Y^{(2)}$, and it follows from (5) that

$$N = N^{(2)}. \quad (29)$$

There are two possibilities for the subgroup N : either $N \neq \{0\}$ or $N = \{0\}$.

a. Assume that $N \neq \{0\}$. Put $W = N^*$. It follows from (29) that the group W contains no elements of order 2. Consider the restriction of equation (3) to N and apply Theorem B to the group W . We get that the restrictions of the characteristic functions $\widehat{\mu}_k(y)$ to N are the characteristic functions of some Gaussian distributions. It follows easily from this that

$$p(\mu_k) = \gamma * m_V * E_{x_k}, \quad k = 1, 2, \quad (30)$$

where $\gamma \in \Gamma^s(X/X_{(2)})$, V is a compact Corwin subgroup of the factor-group $X/X_{(2)}$, $x_k \in X/X_{(2)}$. Moreover, since either $\widehat{\mu}_1(y) \equiv 0$ or $\widehat{\mu}_2(y) \equiv 0$ for $y_1 + Y^{(2)}$, it is not difficult to prove taking into account (30) that at least one of the distributions μ_k is represented in the form

$$\mu_k = \lambda * m_U * E_{z_k},$$

where $\lambda \in \Gamma^s(X)$, U is a compact Corwin subgroup of X , $z_k \in X$.

b. Suppose that $N = \{0\}$. Obviously, it is possible only if X is a compact group. We have $V = A(X/X_{(2)}, N) = X/X_{(2)}$. Taking into account that the equality $N_1 \cap Y^{(2)} = N_2 \cap Y^{(2)} = N \cap Y^{(2)}$ always holds, we get the representation

$$p(\mu_k) = m_{X/X_{(2)}}, \quad k = 1, 2. \quad (31)$$

Moreover, since either $\widehat{\mu}_1(y) \equiv 0$ or $\widehat{\mu}_2(y) \equiv 0$ for $y_1 + Y^{(2)}$, it follows from (31) that $\mu_k = m_X$ at least for one of the distributions μ_k . Let for example, $\mu_1 = m_X$. Returning to the random variables ξ_k , it means that ξ_1 and $2\xi_2$ are identically distributed random variables with distribution m_X .

As a result we obtain the following statement.

Theorem 2. *Let X be a second countable locally compact Abelian group such that X is topologically isomorphic to a group of the form $\mathbf{R}^m \times K$, where $m \geq 0$ and K is a compact Corwin group. Assume also that the group X contains only one element of order 2. Let p be the natural homomorphism $p : X \mapsto X/X_{(2)}$, and $Y = Y^{(2)} \cup (y_1 + Y^{(2)})$ be a decomposition of the group Y with respect to the subgroup $Y^{(2)}$. Assume that ξ_1, ξ_2 are independent random variables with values in X and distributions μ_1, μ_2 such that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent. Then either*

$$\mu_k = \gamma * \epsilon_k * m_V * E_{z_k},$$

where $\gamma \in \Gamma^s(X)$, ϵ_k are signed measures on $X_{(2)}$ such that $\epsilon_1 * \epsilon_2 = E_0$, V is a compact Corwin subgroup of X , $z_k \in X$, $k = 1, 2$, or

$$p(\mu_k) = \gamma * m_V * E_{x_k},$$

where $\gamma \in \Gamma^s(X/X_{(2)})$, V is a compact Corwin subgroup of the factor-group $X/X_{(2)}$, $x_k \in X/X_{(2)}$, $k = 1, 2$, and at least one of the distributions μ_k is represented in the form

$$\mu_k = \lambda * m_U * E_{z_k},$$

where $\lambda \in \Gamma^s(X)$, U is a compact Corwin subgroup of X , $z_k \in X$.

We illustrate Theorem 2 by the following example. Let $X = \mathbf{T}$. Then $Y \cong \mathbf{Z}$. In order not to complicate notation we assume that $Y = \mathbf{Z}$. We have $\mathbf{T}_{(2)} \cong \mathbf{Z}(2)$, and hence there are two possibilities: either $G = \{0\}$ or $G = \mathbf{T}_{(2)}$.

1. $G = \{0\}$. We observe that all compact Corwin subgroups V of the group \mathbf{T} are of the form: either $V = \mathbf{T}$, or V is the subgroup of m th roots of 1, where m is odd, i.e. $V \cong \mathbf{Z}(m)$. If $V = \mathbf{T}$, then by Theorem 2

$$\mu_1 = \mu_2 = m_{\mathbf{T}}.$$

Let $V \cong \mathbf{Z}(m)$. Then by Theorem 2

$$\mu_k = \gamma * \epsilon_k * m_V * E_{z_k}, \quad (32)$$

where $\gamma \in \Gamma^s(\mathbf{T})$, ϵ_k are signed measures on $\mathbf{T}_{(2)}$ such that $\epsilon_1 * \epsilon_2 = E_1$, $z_k \in \mathbf{T}$, $k = 1, 2$. It follows from $V \cong \mathbf{Z}(m)$ that $N = \mathbf{Z}^{(m)}$, and hence (32) implies that the characteristic functions of distributions μ_k are represented in the form

$$\begin{aligned} \hat{\mu}_1(n) &= \begin{cases} \exp\{-\sigma n^2 + int_1\}, & n \in \mathbf{Z}^{(2m)}, \\ \exp\{-\sigma n^2 + int_1 + q\}, & n \in \mathbf{Z}^{(m)} \setminus \mathbf{Z}^{(2m)}, \\ 0, & n \notin \mathbf{Z}^{(m)}, \end{cases} \\ \hat{\mu}_2(n) &= \begin{cases} \exp\{-\sigma n^2 + int_2\}, & n \in \mathbf{Z}^{(2m)}, \\ \exp\{-\sigma n^2 + int_2 - q\}, & n \in \mathbf{Z}^{(m)} \setminus \mathbf{Z}^{(2m)}, \\ 0, & n \notin \mathbf{Z}^{(m)}, \end{cases} \end{aligned}$$

where $\sigma \geq 0$, $t_k, q \in \mathbf{R}$, $k = 1, 2$.

2. $G = \mathbf{T}_{(2)}$. It follows from (29) that $N = \{0\}$. Hence, $V = A(\mathbf{T}/\mathbf{T}_{(2)}, N) = \mathbf{T}/\mathbf{T}_{(2)}$. We get by Theorem 2 $p(\mu_k) = m_{\mathbf{T}/\mathbf{T}_{(2)}}$, $k = 1, 2$, i.e. $\hat{\mu}_1(2n) = \hat{\mu}_2(2n) = 0$, $n \in \mathbf{Z}$, $n \neq 0$. Moreover, either $\hat{\mu}_1(2n+1) = 0$ or $\hat{\mu}_2(2n+1) = 0$, $n \in \mathbf{Z}$. This implies that at least one of the distributions μ_k , say $\mu_1 = m_{\mathbf{T}}$. As regards to the second distribution, we know only that $\hat{\mu}_2(2n) = 0$, $n \in \mathbf{Z}$, $n \neq 0$. Returning to the random variables ξ_k it means that ξ_1 and $2\xi_2$ are identically distributed random variables with the distribution $m_{\mathbf{T}}$.

The obtained description of possible distributions μ_k for the group $X = \mathbf{T}$ is the main content of article [5]. Similarly, it is possible to get from Theorem 2 the description of possible distributions μ_k for the group $\mathbf{R} \times \mathbf{T}$ and for the \mathbf{a} -adic solenoids $\Sigma_{\mathbf{a}}$ found in [11].

Remark 2. Let $X = \mathbf{T}^2$. Then $Y \cong \mathbf{Z}^2$. Denote by $x = (e^{it}, e^{is})$ elements of the group X , and by $y = (m, n) \in \mathbf{Z}^2$ elements of the group Y . We will construct independent random variables ξ_1 and ξ_2 with values in the group X and distributions μ_1 and μ_2 such that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent, and $\mu_k = \gamma * \pi_k$, where $\gamma \in \Gamma^s(X)$, and π_k are signed measures supported in $X_{(4)}$, but not in $X_{(2)}$. Thus, we will show that the statement in Theorem 1 that π_k are signed measures on $(X/G)_{(4)}$, generally speaking can not be strengthened to the statement that π_k are signed measures on $(X/G)_{(2)}$ (compare with Theorem 2).

Take $\sigma > 0$ such that

$$\sum_{(m,n) \in \mathbf{Z}^2, (m,n) \neq (0,0)} e^{-\sigma(m^2+n^2)} < 1. \quad (33)$$

Consider on the group Y the functions

$$l_1(m, n) = \begin{cases} 1, & (m, n) \in \{Y^{(2)}, (1, 0) + Y^{(4)}, (3, 0) + Y^{(4)}, (0, 1) + Y^{(4)}, \\ & (0, 3) + Y^{(4)}, (1, 1) + Y^{(4)}, (3, 3) + Y^{(4)}\}, \\ -1, & (m, n) \in \{(1, 2) + Y^{(4)}, (3, 2) + Y^{(4)}, (2, 1) + Y^{(4)}, \\ & (2, 3) + Y^{(4)}, (1, 3) + Y^{(4)}, (3, 1) + Y^{(4)}\} \end{cases}$$

and

$$l_2(m, n) = \begin{cases} 1, & (m, n) \in \{Y^{(2)}, (1, 0) + Y^{(4)}, (3, 0) + Y^{(4)}, (0, 1) + Y^{(4)}, \\ & (0, 3) + Y^{(4)}, (1, 3) + Y^{(4)}, (3, 1) + Y^{(4)}\}, \\ -1, & (m, n) \in \{(1, 2) + Y^{(4)}, (3, 2) + Y^{(4)}, (2, 1) + Y^{(4)}, \\ & (2, 3) + Y^{(4)}, (1, 1) + Y^{(4)}, (3, 3) + Y^{(4)}\}. \end{cases}$$

Consider on the group X the functions

$$\rho_k(x) = \rho_k(e^{it}, e^{is}) = 1 + \sum_{(m,n) \in \mathbf{Z}^2, (m,n) \neq (0,0)} e^{-\sigma(m^2+n^2)} l_k(m,n) e^{-i(tm+sn)}, \quad k = 1, 2. \quad (34)$$

Inequality (33) implies that $\rho_k(x) > 0$ and it is obvious that

$$\int_X \rho_k(x) dm_X(x) = 1, \quad k = 1, 2.$$

Let μ_k be the distributions on the group X with the densities $\rho_k(x)$ with respect to the Haar distribution m_X , and let ξ_k be independent random variables with values in X and distributions μ_k , $k = 1, 2$. It follows from (34) that

$$\widehat{\mu}_k(m, n) = e^{-\sigma(m^2+n^2)} l_k(m, n), \quad (m, n) \in \mathbf{Z}^2, \quad k = 1, 2. \quad (35)$$

We can check directly that the functions $l_k(m, n)$ satisfy equation (3), and hence the characteristic functions $\widehat{\mu}_k(m, n)$ also satisfy equation (3). Thus, the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent.

By the construction, the functions $l_k(m, n)$ are invariant with respect to the subgroup $Y^{(4)}$, and hence, they define some functions $\widetilde{l}_k(m, n)$ on the factor-group $Y/Y^{(4)} \cong (\mathbf{Z}(4))^2$. By Lemma 3 there exist complex measures π_k on X supported in $A(X, Y^{(4)}) = X_{(4)}$ such that

$$\widehat{\pi}_k(m, n) = l_k(m, n), \quad (m, n) \in \mathbf{Z}^2, \quad k = 1, 2. \quad (36)$$

Since the functions $\widetilde{l}_k(m, n)$ satisfy the condition (26), reasoning as in the proof of Theorem 1 we get that the complex measures π_k actually are signed measures. Since the characteristic functions $\widehat{\pi}_k(m, n)$ are not invariant with respect to the subgroup $Y^{(2)}$, the supports of the signed measures π_k do not contain in $X_{(2)}$. Let γ be a symmetric Gaussian distribution on X with the characteristic function

$$\widehat{\gamma}(m, n) = e^{-\sigma(m^2+n^2)}, \quad (m, n) \in \mathbf{Z}^2. \quad (37)$$

It follows from (35)–(37) that $\mu_k = \gamma * \pi_k$, $k = 1, 2$.

Remark 3. Assume that under conditions of Theorem 1 the independent random variables ξ_1 and ξ_2 are identically distributed, i.e. $\mu_1 = \mu_2 = \mu$. As has been proved in [12], this implies that

$$\mu = \gamma * \pi * m_V * E_x,$$

where $\gamma \in \Gamma^s(X)$, π is a signed measure on $X_{(2)}$ such that $\pi^{*2} = E_0$, V is a compact Corwin subgroup of the group X , $x \in X$. Taking into account Theorem 1 and the example constructing in Remark 2, we see that even we assume that in Theorem 1 $G = \{0\}$, the distributions μ_k have generally speaking factorisation much more complicated than in the case of identically distributed random variables.

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